Math 254B Lecture 22 Notes

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1 Multiplication-Invariant Subsets of $\mathbb{T}$

1.1 Box-counting dimension

Let $k \geq 2$ be an integer. Then consider the map $T_k : \mathbb{T} \to \mathbb{T}$ sending $x \mapsto kx$. Consider a closed subset $X \subseteq \mathbb{T}$ such that $T_k X \subseteq X$. This gives a topological dynamical system $(X, T_k|_X)$ (which we may write (X, T_k)). There is a semi-conjugacy, $\pi : ([k]^{\mathbb{N}}, \sigma) \to (\mathbb{T}, R_k)$ sending $(\omega_i)_i \mapsto \sum_{i \geq 1} \frac{\omega_i}{k^i}$. Then $\pi|_{\pi^{-1}[X]}$ is a semiconjugacy $(\pi^{-1}[X], \sigma) \to (X, T_k)$.

Proposition 1.1. *Under the above conditions,*

$$\underline{\dim}_B(X) = \overline{\dim}_B(X) = \frac{h_{\mathrm{top}}(X, T_k)}{\log(k)}.$$

Proof. Let ρ be usual metric on $\mathbb{T}$. Then

$$\begin{aligned} \rho_n(x, y) &= \max_{0 \leq t \leq n-1} \rho(T^t x, T^t y) \\ &\leq \min\{1, k^n \rho(x, y)\} \end{aligned}$$

and

$$\rho_n(x, y) \geq k^{-1} \min\{1, k^n \rho(x, y)\}.$$

Fix $\delta > 0$. Then

$$\mathrm{cov}_{\delta k^{-n+1}}(X, \rho) \leq \mathrm{cov}_\delta(X, \rho_n) \leq \mathrm{cov}_{\delta k^{-n}}(X, \rho).$$

Then we get

$$\frac{\log(\mathrm{cov}_{\delta k^{-n+1}}(X, \rho))}{\log(\delta) - (n-1)\log(k)},$$

so the liminf of this is equal to $\underline{\dim}_B(X)$. Similarly, the limsup is $\overline{\dim}_B(X)$. But for all $\varepsilon > 0$¡ there is a $\delta > 0$ such that

$$h_{\mathrm{top}} - \varepsilon + o(1) \leq \frac{\log(\mathrm{cov}_\delta(X, \rho_n)}{n} \leq h_{\mathrm{top}} + o(1).$$

$\square$

1.2 Local dimension and Hausdorff dimension

Proposition 1.2. *If $\mu \in P^{T_k}(X)$ is atomless and ergodic, then*

$$\underline{\dim}(\mu) = \overline{\dim}(\mu) = \frac{h(\mu, T_k)}{\log(k)}$$

Proof. Let $\alpha_i : X \to [k]$ be such that $x = 0.\alpha_1(x)\alpha_2(x)\alpha_3(x)\cdots$ with the convention that $\{\alpha_i = i\} = [i/k, (i+1)/k)$. Then α_1 generates this, so $h(\mu, T_k) = H(\mu, T_k, \alpha_1) = h$. Now consider $\alpha_{[1;n]} : X \to [k]^n$. The level sets are $[i/k^n, (i+1)/k^n)$ for $0 \le i \le k^n - 1$. Call this partition $\mathcal{D}_n$. Let $D_n(x)$ be the cell of $\mathcal{D}_n$ containing x. By the Shannon-McMillan-Breiman theorem, μ-a.e. x satisfies $\mu(D_n(x)) = e^{-hn+o(n)}$.

1. For all x, we have $B_{k^{-n+1}}(x) \supseteq D_n(x)$, so

$$\mu(B_{k^{-n+1}}(x)) = \mu(D_n(x)) \ge e^{-hn-o(n)} = k^{-(h/\log(k))n+o(n)} \quad \text{a.s.}$$

2. Suppose $\mu(A) > 0$. Fix $\varepsilon > 0$, and let $B_m = \{x : \mu(D_n(x)) \le e^{-hn+\varepsilon n} \forall n \ge m\}$. Then $\mu(\bigcup_m B_m) = 1$, there is an m such that $\mu(A \cap B_m) > 0$. Define $\nu := \mu(\cdot \mid A \cap B_m)$. Now consider $x \in A \cap B_m$ and a ball $B_{k^{-n-1}}(x) \subseteq D_n(x) \cup D'$, where D'_n is a neighbor interval at the same level of $D_n(x)$. So for $n \ge m$,

$$\nu(B_{k^{-n-1}}(x)) \le \underbrace{\nu(D_n(x))}_{\le O(1)\mu(D_n(x))} + \nu(D'_n) \le O(1)e^{-hn+\varepsilon n} + O(1)e^{hn+\varepsilon n},$$

where $\nu(D'_n)$ either equals 0 or is $\le O(1)e^{hn+\varepsilon n}$ in case $D'_n \cap b_m \ne \varnothing$. So for all $x \in A \cap B_m$, we have $\nu(B_{k^{-n-1}}(x)) \le O(1)k^{-(n-\varepsilon)/\log(k)\cdot h}$. $\square$

Corollary 1.1. *If $X \subseteq \mathbb{T}$ is closed and T_k-invariant, then*

$$\dim(X) = \frac{h_{\text{top}}(X, T_k)}{\log(k)}.$$

Proof. We know $\dim(X) \le \underline{\dim}_B(X)$. For the reverse, we get

$$\dim(X) \ge \sup_{\mu \in P_e^{T_k}(X)} \overline{\dim}(\mu) = \sup_{\mu \in P_e^{T_k}(X)} \frac{h(\mu, T_k)}{\log(k)}.$$

The result follows by the variational principle. $\square$

Remark 1.1. We know that by the variational principle,

$$h_{\text{top}}(X, T_k) = \sup_\mu h(\mu, T_k).$$

The statement

$$\dim(X) = \sup_\mu \underline{\dim}(\mu).$$

should be viewed analogously.

1.3 Dimension of occurrences of digits

Proposition 1.3. *Fix* $p = (p_0, \ldots, p_{k-1}) \in P([k])$*, and let*

$$F(p) = \left\{ x \in \mathbb{T} : \lim_{n\to\infty} \frac{|\{t \le n : \alpha_1(x) = t\}}{n} = p_j \; \forall j \right\}.$$

Then

$$\dim(F(p)) = \frac{H(p)}{\log(k)}.$$

Proof. Let $\mu_P = \pi_*(p^{\times\mathbb{N}}) \in P^{T_k}(\mathbb{T})$. Now $x \in F(p)$ iff $\mu_P(D_n(x)) = e^{-h(p)n+o(n)}$. Use Billingsley's lemma.[1] ☐

The picture does not always work out perfectly, however.

Example 1.1. Consider $T = T_2 \times T_3 : \mathbb{T}^2 \to \mathbb{T}^2$. What is $\dim(X)$ for $TX \subseteq X$? In general, we cannot describe this in terms of dynamical properties.

[1]This was actually the original use of Bllingsley's lemma.